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The fuzzy sphere: from the uncertainty relation to the stereographic projection

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Abstract: On the fuzzy sphere, no state saturates simultaneously all the Heisenberg uncertainties. We propose a weaker uncertainty for which this holds. The family of states so obtained is physically motivated because it encodes information about positions in this fuzzy context. In particular, these states realize in a natural way a deformation of the stereographic projection. Surprisingly, in the large j limit, they reproduce some properties of the ordinary coherent states on the non commutative plane.

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Contents

1. Introduction

The idea that at very small scale the physical structure of space time may display non trivial features has received wide interest. In particular non commutative theories have been under scrutiny in the last years. One of the key tools in this framework is the star product, which allows the treatment of these theories using commuting variables. The first star product concerned phase space [1, 2, 3]; it is readily transfered to the non commutative plane composed of two position operators whose commutator is a constant.

Among the methods used to build star products, the one based on coherent states is especially interesting since associativity is then automatically assured. This is a consequence of the fact that coherent states form an over complete basis on the Hilbert space on which the algebra of quantum operators act.

The terms "coherent states" and "generalized coherent states" are used in the literature following the works of Perelomov and Berezin for example. These notions are defined in connection with irreducible representations of Lie groups and quotient spaces by these groups. The states appearing here arise differently: we generalize a different property of the system of coherent states on the non commutative plane.

On the non commutative plane, coherent states are obtained in a straightforward way. They saturate the bound of the Heisenberg uncertainty and this is equivalent to the fact that they are eigenstates of the destruction operator [4, 5]. On the fuzzy sphere, the situation is quite different. Firstly, there is no state saturating all the uncertainties. Secondly, the identification of a destruction operator is not straightforward; one possibility is the use of the stereographic projection to obtain a deformed creation-destruction algebra. The problem then is the difference between the dimensions of the Fock spaces; one usually wishes to use finite dimensional representations of SU(2) for the fuzzy sphere but the Fock space on which the deformed creation-destruction algebra acts is infinite dimensional. One of the proposals to tackle this problem was trough a truncature procedure [6].

We propose here another generalization which starting point is not a deformed destruction operator but an enlarged Heisenberg uncertainty. The states saturating the associated bounds are called "generalized" squeezed states. Another approach, relying on a functional [7] and exploiting an idea introduced in another context [8], led to technical problems. The procedure followed here is not as general and ambitious as the ones developed in [9]–[12]. Like in [6] and contrary to [11, 12], we expect the induced star product to be expressed in terms of two independent rather than three dependent coordinates. The derivation and the use of the associated star product to study quantum field theory like in [13]–[25] and especially the Seiberg-Witten map [26] are our final goal [27].

This paper is organized as follows. In the second section we summarize the construction of coherent states on the fuzzy sphere developed in [6] and discuss its differences with the procedure used for the non commutative plane. The third section is devoted to the derivation of what we call generalized squeezed states on a two dimensional space. In section 4 we show that while the Heisenberg uncertainties on the non commutative sphere can not be saturated simultaneously, the ones we derived in the previous section just do that. We obtain that the scalar product of two generalized squeezed states reproduces, in the $j \to \infty$ limit, the corresponding expression on a non commutative plane. In section 5, we show that the mean values of the position operators on these states lead to the stereographic projection.

We emphasize that the appearance of the stereographic projection obtained here is of a different nature compared to the ones of [6, 28]. In these two works, the stereographic projection is introduced from the start to define a pair of creation-destruction like operators from the three quantum ones defining the fuzzy sphere. On the contrary, we begin with an uncertainty relation which is weaker than the Heisenberg's and find that it leads to a family of states parameterized by a complex number in a way which brings in the stereographic map.

2. Coherent states: the non commutative plane versus the fuzzy sphere

The non commutative plane is defined by the relations [29]

$$[\hat{x}_j, \hat{x}_k] = i\theta\,\epsilon_{j,k}\,, \qquad \theta > 0\,, \qquad k, j = 1, 2\,. \tag{2.1}$$

The coherent states are defined as eigenvalues of a destruction operator and saturate the Heisenberg uncertainty $\Delta x_1 \Delta x_2 = \theta/2$.

The situation of the fuzzy sphere is far from being so simple. It is a matrix model defined by the following relations [30, 31]:

$$[\hat{x}_k, \hat{x}_l] = \frac{iR}{\sqrt{j(j+1)}}\epsilon_{klm}\hat{x}_m\,, \qquad \delta^{lk}\hat{x}_l\hat{x}_k = R^2\,,$$
$$\text{with } j \text{ integer or half-integer and } k, l, m = 1, 2, 3\,. \tag{2.2}$$

We will use rescaled variables such that the radius R can be given the value one. There is no state saturating simultaneously all the Heisenberg uncertainties [7]

$$\Delta x_1 \Delta x_2 = \frac{1}{2}\frac{1}{\sqrt{j(j+1)}}|\langle \hat{x}_3\rangle| , \qquad \Delta x_2 \Delta x_3 = \frac{1}{2}\frac{1}{\sqrt{j(j+1)}}|\langle \hat{x}_1\rangle| ,$$
$$\Delta x_3 \Delta x_1 = \frac{1}{2}\frac{1}{\sqrt{j(j+1)}}|\langle \hat{x}_2\rangle| ; \tag{2.3}$$

we will come back to this point in the fourth section. As a consequence, one has to resort to other criteria to define coherent states in this context. One way to tackle this problem is based on the construction of a deformation of the creation-destruction operators [6]. This is summarized below.

One uses the stereographic projection to define the operators

$$z = (\hat{x}_1 - i\hat{x}_2)(1-\hat{x}_3)^{-1} , \qquad z^+ = (1-\hat{x}_3)^{-1}(\hat{x}_1 + i\hat{x}_2) \tag{2.4}$$

which obey the commutation relation

$$[z, z^+] = F(zz^+) , \tag{2.5}$$

where

$$F(zz^+) = \alpha\chi\left[1 + |z|^2 - \frac{1}{2}\chi\left(1 + \frac{\alpha}{2}|z|^2\right)\right] , \qquad \chi = \frac{2}{\alpha}\left[1 + \frac{\alpha}{2\xi} - \sqrt{\frac{1}{\xi} + \left(\frac{\alpha}{2\xi}\right)^2}\,\right] ,$$
$$\xi = 1 + \alpha|z|^2 , \qquad \text{and} \qquad \alpha = \frac{1}{\sqrt{j(j+1)}} . \tag{2.6}$$

Thanks to eq. (2.5), there exists a function f which relates the operators z, z^+ to the usual creation and destruction operators $\hat{a}^+, \hat{a}$:

$$z = f(\hat{a}^+\hat{a} + 1)\hat{a} . \tag{2.7}$$

The generalized coherent states $|\zeta\rangle$ are taken to be the eigenstates of the destruction operator z and read

$$|\zeta\rangle = N(|\zeta|^2)^{-1/2} \qquad \exp\left[\zeta f^{-1}(\hat{a}^+\hat{a})\hat{a}^+\right] f^{-1}(\hat{a}^+\hat{a})|0\rangle , \tag{2.8}$$

the function $N(|\zeta|^2)^{-1/2}$ enforcing the normalization of the wave function. The vacuum $|0\rangle$ is annihilated by $\hat{a}$.

It should be stressed that the map between the couples of operators z, z^+ and $\hat{a}, \hat{a}^+$ is singular since the second pair act on an infinite Fock space while the first acts on a finite dimensional vector space. This was taken into account by restricting, for a fixed j, the expansion of eq. (2.8) to the component $|2j\rangle$:

$$|\zeta, j\rangle = N_j(|\zeta|^2)^{-1/2} \sum_{n=0}^{2j} \frac{\zeta^n}{\sqrt{n!}[f_j(n)]!}|n\rangle , \qquad N_j(x) = \sum_{n=0}^{2j} \frac{x^n}{n!([f_j(n)]!)^2} ,$$
$$f_j(n) = \frac{\sqrt{2j-n+1}}{\sqrt{j(j+1)} + j - n} , \qquad \text{where } [f(n)]! = f(n)f(n-1)\ldots f(0) . \tag{2.9}$$

The states obtained in this way are not eigenstates of the operator z anymore.

The complications about coherent states on the fuzzy sphere come from the fact that, contrary to the non commutative plane, the Heisenberg uncertainties given in eq. (2.3) can not be saturated simultaneously. If one relies on a mapping from z, z^+ to the ordinary creation-destruction operator, one has to face the discrepancy between the dimensions of the spaces involved. If one cuts the sum at a given index, the states obtained are not exact eigenstates of the operator z.

The main point of this work is that there is a natural extension of the Heisenberg uncertainty which allows one to define "generalized squeezed states" for the fuzzy sphere in a way which is more or less similar to what one does for the quantum plane. The dimensionality problem pointed above is absent in this approach.

In brief, we shall show that having two operators $\hat{x}, \hat{p}$, the most general uncertainty relation implying the mean values of expressions at most quadratic in the operators is

$$(\Delta x)^2(\Delta p)^2 \geq \frac{1}{4}\left[(\langle\{\hat{x},\hat{p}\}\rangle - 2\langle\hat{p}\rangle\langle\hat{x}\rangle)^2 + |\langle[\hat{x},\hat{p}]\rangle|^2\right]. \tag{2.10}$$

This relation involves not only the commutator but also the anti commutator; the Heisenberg uncertainty can be deduced from it. For ordinary coherent states, the first quantity under square brackets on the right side vanishes and the Heisenberg bound is attained. We shall show that the equations

$$\begin{aligned}
(\Delta x_1)^2(\Delta x_2)^2 &= \frac{1}{4}\left[(\langle\{\hat{x}_1,\hat{x}_2\}\rangle - 2\langle\hat{x}_1\rangle\langle\hat{x}_2\rangle)^2 + |\langle[\hat{x}_1,\hat{x}_2]\rangle|^2\right], \\
(\Delta x_2)^2(\Delta x_3)^2 &= \frac{1}{4}\left[(\langle\{\hat{x}_2,\hat{x}_3\}\rangle - 2\langle\hat{x}_2\rangle\langle\hat{x}_3\rangle)^2 + |\langle[\hat{x}_2,\hat{x}_3]\rangle|^2\right], \\
(\Delta x_3)^2(\Delta x_1)^2 &= \frac{1}{4}\left[(\langle\{\hat{x}_3,\hat{x}_1\}\rangle - 2\langle\hat{x}_3\rangle\langle\hat{x}_1\rangle)^2 + |\langle[\hat{x}_3,\hat{x}_1]\rangle|^2\right],
\end{aligned} \tag{2.11}$$

are compatible on the fuzzy sphere. The states obeying them simultaneously are what we shall call "generalized squeezed states". We will show that they are parameterized by a point on the complex plane and possess other interesting properties.

3. Squeezed states: the $2D$ phase space revisited

To begin, let us consider a two dimensional phase space whose quantum operators are denoted $\hat{x}, \hat{p}$. We will work on a linear combination of the two operators supplemented by a term proportional to the identity:

$$\hat{\Theta}_\lambda = \hat{x} + (\lambda_1 + i\lambda_2)\hat{p} + (\lambda_3 + i\lambda_4)I\,, \tag{3.1}$$

the λ_i being real constants. In the derivation of the Heisenberg inequality, one assumes for example $\lambda_1 = 0$. We will recover the known result as a particular case.

For any state $|\psi\rangle$ and any real λ_i, the following quantity is positive or null:

$$||\hat{\Theta}_\lambda|\psi\rangle||^2 \geq 0\,. \tag{3.2}$$

If one fixes the state $|\psi\rangle$, the preceding function is a second degree polynomial in the λ_i:

$$\begin{aligned}\varphi(\lambda_1,\lambda_2,\lambda_3,\lambda_4) &= \langle\psi|\hat\Theta^+_\lambda\hat\Theta_\lambda|\psi\rangle\\ &= (\lambda_1^2+\lambda_2^2)\langle\hat p^2\rangle + (\lambda_3^2+\lambda_4^2) + 2(\lambda_1\lambda_3+\lambda_2\lambda_4)\langle\hat p\rangle + \\ &\quad +\lambda_1 A_{xp} + \lambda_2 C_{xp} + 2\lambda_3\langle\hat x\rangle + \langle\hat x^2\rangle\,. \end{aligned}\tag{3.3}$$

We have used the notations

$$A_{xp} = \langle\hat x\hat p + \hat p\hat x\rangle\,;\qquad C_{xp} = i(\langle\hat x\hat p - \hat p\hat x\rangle) \tag{3.4}$$

to simplify future formulas and assumed a normalization to unity: $\langle\psi|\psi\rangle = 1$. Due to hermiticity conditions, the mean values A_{xp} and C_{xp} are real. Our polynomial can be diagonalized on its real variables:

$$\begin{aligned}\varphi(\lambda_1,\lambda_2,\lambda_3,\lambda_4) &= \langle\hat p^2\rangle\left[\boldsymbol{\lambda_1} + \left(\frac{1}{2}A_{xp} + \langle\hat p\rangle\boldsymbol{\lambda_3}\right)\frac{1}{\langle\hat p^2\rangle}\right]^2 + \\ &\quad+\langle\hat p^2\rangle\left[\boldsymbol{\lambda_2} + \left(\frac{1}{2}C_{xp} + \langle\hat p\rangle\boldsymbol{\lambda_4}\right)\frac{1}{\langle\hat p^2\rangle}\right]^2 + \\ &\quad+\frac{(\Delta p)^2}{\langle\hat p^2\rangle}\left[\boldsymbol{\lambda_3} + \frac{2\langle\hat x\rangle\langle\hat p^2\rangle - A_{xp}\langle\hat p\rangle}{2(\Delta p)^2}\right]^2 + \frac{(\Delta p)^2}{\langle\hat p^2\rangle}\left[\boldsymbol{\lambda_4} - \frac{C_{xp}\langle\hat p\rangle}{2(\Delta p)^2}\right]^2 + \\ &\quad+\left[\langle\hat x^2\rangle - \frac{1}{4}\frac{A_{xp}^2}{\langle\hat p^2\rangle} - \frac{1}{4}\frac{C_{xp}^2}{\langle\hat p^2\rangle} - \frac{1}{4}\frac{(2\langle\hat x\rangle\langle\hat p^2\rangle - A_{xp}\langle\hat p\rangle)^2}{\langle\hat p^2\rangle(\Delta p)^2} - \frac{1}{4}\frac{C_{xp}^2\langle\hat p\rangle^2}{\langle\hat p^2\rangle(\Delta p)^2}\right].\end{aligned}\tag{3.5}$$

The content of the four first brackets in this formula form a complete set of independent variables built from the λ_i so that the positivity of our polynomial implies $(\Delta p)^2, \langle\hat p^2\rangle > 0$ which are trivially verified. The only valuable inequality we infer is that

$$\min\varphi = \langle\hat x^2\rangle - \frac{1}{4}\frac{A_{xp}^2}{\langle\hat p^2\rangle} - \frac{1}{4}\frac{C_{xp}^2}{\langle\hat p^2\rangle} - \frac{1}{4}\frac{(2\langle\hat x\rangle\langle\hat p^2\rangle - A_{xp}\langle\hat p\rangle)^2}{\langle\hat p^2\rangle(\Delta p)^2} - \frac{1}{4}\frac{C_{xp}^2\langle\hat p\rangle^2}{\langle\hat p^2\rangle(\Delta p)^2} \geq 0\,, \tag{3.6}$$

since the polynomial is positive or null for any choice of the variables λ_i. This minimum is attained for the following values of these variables:

$$\begin{aligned}\bar\lambda_1 &= -\frac{1}{2\langle\hat p^2\rangle}(A_{xp} - 2\langle\hat p\rangle\langle\hat x\rangle)\left(1 + \frac{\langle\hat p\rangle^2}{2(\Delta p)^2}\right), &\bar\lambda_2 &= -\frac{1}{2}\frac{C_{xp}}{(\Delta p)^2}\\ \bar\lambda_3 &= -\langle\hat x\rangle - \frac{\langle\hat p\rangle}{2(\Delta p)^2}\left(-A_{xp} + 2\langle\hat p\rangle\langle\hat x\rangle\right), &\bar\lambda_4 &= \frac{1}{2}\frac{\langle\hat p\rangle}{(\Delta p)^2}C_{xp}\,.\end{aligned}\tag{3.7}$$

The inequality displayed in eq. (3.6) involves all momenta of first and second degree in the phase space variables. It contains the anticommutator and the commutator mean values A_{xp} and C_{xp}. The apparent lack of symmetry between the position and the momentum mean values $\langle\hat x\rangle, \langle\hat p\rangle$ is linked to the fact that the operator of eq. (3.1) was not symmetric in the two operators; moreover we broke the symmetry between the λ_i in our diagonalization.

JHEP05(2004)052

Actually, this inequality can be rewritten in a closer and more symmetric form. First, let us consider its second and fourth terms which are the only ones containing the anti commutator. Putting them on the same denominator, one obtains

$$-\frac{1}{4\langle\hat{p}^2\rangle(\Delta p)^2}\left[((\Delta p)^2+\langle\hat{p}\rangle^2)A_{xp}^2-2\langle\hat{x}\rangle\langle\hat{p}^2\rangle\langle\hat{p}\rangle A_{xp}+4\langle\hat{x}\rangle^2\langle\hat{p}^2\rangle^2\right]=$$

$$=\frac{1}{4(\Delta p)^2}\left[(A_{xp}-2\langle\hat{x}\rangle\langle\hat{p}\rangle)^2+4\langle\hat{x}\rangle^2(\Delta p)^2\right]. \tag{3.8}$$

In the same way, the third and fifth terms which contain the anticommutator can be rewritten as

$$\frac{1}{4(\Delta p)^2}C_{xp}^2\,. \tag{3.9}$$

The inequality obtained above can now be rewritten in the more compact form

$$(\Delta x)^2(\Delta p)^2\geq\frac{1}{4}\left((A_{xp}-2\langle\hat{p}\rangle\langle\hat{x}\rangle)^2+C_{xp}^2\right). \tag{3.10}$$

The minimum of the function φ solely depends on the state $|\psi\rangle$ as can readily be seen in (3.6). We can go one step further and ask which states make the minimum of φ attain its lowest i.e zero value. Such a state will obviously satisfy the equation

$$\hat{\Theta}_{\bar{\lambda}}|\psi\rangle=0 \tag{3.11}$$

and the equality will be attained in eq. (3.10), the parameters $\bar{\lambda}_i$ being related to the mean values of the state by eqs. (3.7).

From this we can derive the Heisenberg uncertainty relation

$$\Delta x\Delta p\geq\frac{1}{2}|C_{xp}|\,; \tag{3.12}$$

when it is saturated, the following equality holds

$$\tilde{A}_{xp}=A_{xp}-2\langle\hat{p}\rangle\langle\hat{x}\rangle=\langle\hat{x}\hat{p}+\hat{p}\hat{x}\rangle-2\langle\hat{p}\rangle\langle\hat{x}\rangle=0\,. \tag{3.13}$$

This relation can be used to eliminate A_{xp} in eq. (3.7); one then obtains the simpler expressions

$$\bar{\lambda}_1=0\,,\qquad\bar{\lambda}_3=-\langle\hat{x}\rangle\,; \tag{3.14}$$

eq. (3.11) then assumes the well known form

$$\left(\hat{x}-\langle\hat{x}\rangle+\frac{C_{xp}}{2(\Delta p)^2}(\hat{p}-\langle\hat{p}\rangle)\right)|\psi\rangle=0\,. \tag{3.15}$$

The main result here is given in eq. (3.10) and the conclusion that

$$\Delta x\Delta p=\frac{1}{2}|\langle\hat{x}\hat{p}-\hat{p}\hat{x}\rangle|\Longleftrightarrow\langle\hat{x}\hat{p}+\hat{p}\hat{x}\rangle-2\langle\hat{p}\rangle\langle\hat{x}\rangle=0\,. \tag{3.16}$$

JHEP05(2004)052

This statement can be verified in the ordinary theory. The commutation relation is then given by $[\hat{x},\hat{p}] = i$. Working in the momentum representation, the solution of eq. (3.15) has the well known gaussian form [32]

$$\psi = \left[\exp\left(-\frac{b_1^2}{2a}\right)\sqrt{-\frac{\pi}{2a}}\right]^{-1/2} e^{(ap^2+(b_1+ib_2)p)} . \tag{3.17}$$

One has

$$\langle\hat{x}\rangle = -b_2\,, \qquad \langle\hat{p}\rangle = -\frac{b_1}{2a}\,, \qquad \langle\hat{p}\hat{x}\rangle = -\frac{i}{2} + \frac{b_1 b_2}{2a}\,, \qquad \langle\hat{x}\hat{p}\rangle = \frac{i}{2} + \frac{b_1 b_2}{2a}\,, \tag{3.18}$$

so that eq. (3.16) is verified. The situation is the same in the one dimensional K.M.M [33] model which is defined by the commutation rules $[\hat{x},\hat{p}] = i(1+\hat{p}^2)$. A momentum representation exists in which the action of the operators and the scalar product defining the Hilbert space are given by

$$\hat{x} = i(1+p^2)\partial_p\,, \qquad \hat{p} = p\,, \qquad \langle\phi|\psi\rangle = \int dp\frac{1}{1+p^2}\phi^*(p)\psi(p)\,. \tag{3.19}$$

This theory admits a minimal uncertainty in length: $\Delta x \geq 1$. Let us verify the left part of eq. (3.16) for the most important solutions of eq. (3.15) in this theory i.e those which exhibit the minimal uncertainty in length:

$$\psi(p) = \sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{1+p^2}}\exp\left(-i\xi\arctan\left(p\right)\right). \tag{3.20}$$

For these states one has $\langle\hat{p}\rangle = 0, \langle\hat{x}\rangle = \xi$ and

$$\langle\hat{x}\hat{p}+\hat{p}\hat{x}\rangle = \int dp\frac{1}{1+p^2}\frac{2}{\pi}\left(\frac{-ip^2+2\xi p+i}{1+p^2}\right) = 0\,, \tag{3.21}$$

so that eq. (3.16) is verified.

At this level, the natural question which arises from our analysis is the existence (or not) of theories in which the term containing the anticommutator may play a role in the uncertainty relation. This may happen if the only normalizable states obeying eq. (3.11) correspond to non vanishing λ_1. This excludes models in which the commutation relations are of the form $[\hat{x},\hat{p}] = if(\hat{p})$ since in this case the relevant states read, in momentum space:

$$\psi(p) = N\exp\left(-\int_0^p d\xi\frac{(\lambda_2\xi+\lambda_4)}{f(\xi)}\right)\exp\left(i\int_0^p d\xi\frac{(\lambda_1\xi+\lambda_3)}{f(\xi)}\right). \tag{3.22}$$

Their normalization requires the finiteness of the integral

$$\langle\psi|\psi\rangle = N^2\int_{-\infty}^{\infty}\frac{dp}{f(p)}\exp\left(-2\int_0^p d\xi\frac{(\lambda_2\xi+\lambda_4)}{f(\xi)}\right) \tag{3.23}$$

which does not depend on λ_1.

The models in which the deformation of the commutation relations is entirely embodied in a function of the momentum are not the only ones which have been studied in the literature. Among the proposals which do not fit in this category is for example the q deformation of quantum mechanics which is defined by the following equations [34]:

$$q^{\frac{1}{2}}\hat{x}\hat{p} - q^{-\frac{1}{2}}\hat{p}\hat{x} = i\hat{\Lambda}\,, \qquad \hat{\Lambda}\hat{p} = q\hat{p}\hat{\Lambda}\,, \qquad \hat{\Lambda}\hat{x} = q^{-1}\hat{x}\hat{\Lambda}\,. \tag{3.24}$$

The argument we gave above can not be applied here. One needs a closer analysis to see if the new term plays any role in such a model. We show in the next section that it proves to be important on the fuzzy sphere.

The reasoning followed here can be extended to higher dimensions. A partial motivation for such a treatment and an illustration in the three variables case is given in the appendix.

4. Generalized squeezed states on the fuzzy sphere

The saturation of the Heisenberg uncertainties (eq. (2.3)) related to the pairs of non commuting variables translate into the formulas [7]

$$\begin{aligned} \hat{m}_{jk}|\psi\rangle &= 0 \qquad \text{with } \hat{m}_{12} = \hat{x}_1 + \mu_{12}\hat{x}_2 + \tau_{12}\,, \qquad \hat{m}_{23} = \hat{x}_2 + \mu_{23}\hat{x}_3 + \tau_{23}\,, \\ \hat{m}_{31} &= \hat{x}_3 + \mu_{31}\hat{x}_1 + \tau_{31}\,, \end{aligned} \tag{4.1}$$

where the μ_{jk} are pure imaginary while the τ_{jk} have real and imaginary parts(see eq. (3.15)). We have adopted simplified notations in this section; writing μ_{jk} rather than $\bar{\mu}_{jk}$. No confusion is possible at this stage, contrary to the previous section.

Considering the following combinations of these equations

$$\begin{aligned} (\mu_{31}[\hat{m}_{12},\hat{m}_{23}] + [\hat{m}_{23},\hat{m}_{31}])|\psi\rangle &= 0\,, \qquad (\mu_{23}[\hat{m}_{31},\hat{m}_{12}] + [\hat{m}_{12},\hat{m}_{23}])|\psi\rangle = 0\,, \\ (\mu_{12}[\hat{m}_{23},\hat{m}_{31}] + [\hat{m}_{31},\hat{m}_{12}])|\psi\rangle &= 0\,, \end{aligned} \tag{4.2}$$

one obtains

$$(1+\mu_{12}\mu_{23}\mu_{31})\hat{x}_1|\psi\rangle = 0\,, \qquad (1+\mu_{12}\mu_{23}\mu_{31})\hat{x}_3|\psi\rangle = 0\,, \qquad (1+\mu_{12}\mu_{23}\mu_{31})\hat{x}_2|\psi\rangle = 0\,. \tag{4.3}$$

Can the three Heisenberg inequalities be saturated simultaneously? Only two cases may lead to that situation:

- The first possibility is
$$\hat{x}_k|\psi\rangle = 0\,, \qquad k=1,2,3\,, \tag{4.4}$$
but then the second part of eq. (2.2) is violated .

- The remaining possibility
$$(1+\mu_{12}\mu_{23}\mu_{31}) = 0 \tag{4.5}$$
is ruled out by the fact that the quantities μ_{jk} are purely imaginary; it can be rewritten as
$$\frac{\langle\hat{x}_1\rangle\langle\hat{x}_2\rangle\langle\hat{x}_3\rangle}{(\Delta x_1)^2(\Delta x_2)^2(\Delta x_3)^2} = -8i\,. \tag{4.6}$$

Let us now turn to the enlarged uncertainty relations. Their saturation is displayed in eq. (2.11). The equation defining these "squeezed" states is the same as eq. (4.1) but now the coefficients μ_{jk} will have real and imaginary parts:

$$\mu_{12} = \lambda_1 + i\lambda_2\,, \qquad \mu_{23} = \lambda_3 + i\lambda_4\,, \qquad \mu_{31} = \lambda_5 + i\lambda_6\,. \tag{4.7}$$

The relation displayed in eq. (4.5), which is necessary for the associated three uncertainties to be saturated simultaneously, can be rewritten as

$$1+(\lambda_1\lambda_3-\lambda_2\lambda_4)\lambda_5-(\lambda_1\lambda_4+\lambda_2\lambda_3)\lambda_6 = 0\,, \qquad (\lambda_1\lambda_3-\lambda_2\lambda_5)\lambda_6+(\lambda_1\lambda_4+\lambda_2\lambda_3)\lambda_5 = 0\,. \tag{4.8}$$

Not taking into account the terms $\tilde{A}_{x_i,x_j}$(defined in eq. (3.13)) amounts to impose $\lambda_1 = \lambda_3 = \lambda_5 = 0$ which transforms the first of the previous equations into the contradiction $1 = 0$. The states we are looking for are in the kernels of the three operators m_{jk} but with the coefficients $\mu_{j,k}$ having non vanishing real parts.

For the remaining part of this work, we will adopt, for the fuzzy sphere, unitary representations of dimensions $(2j+1)(2j+1)$:

$$\begin{aligned}
\hat{x}_k &= \frac{1}{\sqrt{j(j+1)}}\hat{J}_k\,, \qquad \hat{J}_1 = \frac{1}{2}(\hat{J}_+ + \hat{J}_-)\,, \qquad \hat{J}_2 = -\frac{i}{2}(\hat{J}_+ - \hat{J}_-)\,,\\
\hat{J}_+|m\rangle &= \sqrt{(j-\mu)(j+m+1)}|m+1\rangle\,, \qquad \hat{J}_-|m\rangle = \sqrt{(j+\mu)(j-m+1)}|m-1\rangle\,,\\
\hat{J}_3|m\rangle &= m|m\rangle\,.
\end{aligned} \tag{4.9}$$

We now solve eq. (4.1) where the parameters μ_{jk} have real and imaginary parts. For a fixed j, the state we are looking for can be written as

$$|Y\rangle = \sum_{n=-j}^{j} Y_n|n\rangle\,. \tag{4.10}$$

Each condition $\hat{m}_{jk}|Y\rangle = 0$ leads to three "equations": the components of $|-j\rangle, |j\rangle$ and $|n\rangle$ for $-j+1 \leq n \leq j-1$. For example,

$$\begin{aligned}
\hat{m}_{12}|Y\rangle &= \left[\frac{1}{2}\alpha\left((1-i\,\mu_{12})\hat{J}_+ + (1+i\,\mu_{12})\hat{J}_-\right) + \tau_{12}\right]|Y\rangle\\
&= \frac{1}{2}\alpha(1-i\,\mu_{12})\left(Y_{j-1}\sqrt{2j}\,|j\rangle + \sum_{n=-j+1}^{j-1} Y_{n-1}\sqrt{(j+n)(j-n+1)}\,|n\rangle\right) +\\
&\quad +\frac{1}{2}\alpha(1+i\,\mu_{12})\left(Y_{-j+1}\sqrt{2j}\,|-j\rangle + \sum_{n=-j+1}^{j-1} Y_{n+1}\sqrt{(j-n)(j+n+1)}\,|n\rangle\right) +\\
&\quad +\tau_{12}\left(Y_{-j}|-j\rangle + Y_j|j\rangle + \sum_{n=-j+1}^{j-1} Y_n|n\rangle\right),
\end{aligned} \tag{4.11}$$

where $\alpha = 1/\sqrt{j(j+1)}$. We end up with the following "nine" relations:

$$\frac{Y_{-j+1}}{Y_{-j}} = -\frac{2\tau_{12}}{\alpha(1+i\,\mu_{12})\sqrt{2j}}\,, \tag{4.12}$$

JHEP05(2004)052

$$
\begin{aligned}
\frac{Y_j}{Y_{j-1}} &= -\frac{\alpha(1-i\,\mu_{12})\sqrt{2j}}{2\tau_{12}}\,, && (4.13)\\
Y_{n+1}+\frac{2\tau_{12}}{\alpha(1+i\,\mu_{12})\sqrt{(j+n+1)(j-n)}}Y_n+ & \\
+\frac{1-i\,\mu_{12}}{1+i\,\mu_{12}}\frac{\sqrt{(j-n+1)(j+n)}}{\sqrt{(j+n+1)(j-n)}}Y_{n-1} &= 0\,, && (4.14)\\
\frac{Y_{-j+1}}{Y_{-j}} &= \frac{2(-\tau_{23}+\alpha\,\mu_{23}\,j)}{i\,\alpha\sqrt{2j}}\,, && (4.15)\\
\frac{Y_j}{Y_{j-1}} &= \frac{i\,\alpha\,\sqrt{2j}}{2(\alpha\,\mu_{23}j+\tau_{23})}\,, && (4.16)\\
Y_{n+1}-\frac{2\,i\,(\alpha\,\mu_{23}n+\tau_{23})}{\alpha\sqrt{(j+n+1)(j-n)}}Y_n- & \\
-\frac{\sqrt{(j-n+1)(j+n)}}{\sqrt{(j+n+1)(j-n)}}Y_{n-1} &= 0\,, && (4.17)\\
\frac{Y_{-j+1}}{Y_{-j}} &= \frac{2(\alpha\,j-\tau_{31})}{\alpha\,\mu_{31}\,\sqrt{2j}}\,, && (4.18)\\
\frac{Y_j}{Y_{j-1}} &= -\frac{\alpha\,\mu_{31}\,\sqrt{2j}}{2(\alpha\,j+\tau_{31})}\,, && (4.19)\\
Y_{n+1}+\frac{2\,(\alpha\,n+\tau_{31})}{\alpha\,\mu_{31}\,\sqrt{(j+n+1)(j-n)}}Y_n+ & \\
+\frac{\sqrt{(j-n+1)(j+n)}}{\sqrt{(j+n+1)(j-n)}}Y_{n-1} &= 0\,. && (4.20)
\end{aligned}
$$

As this system has more equations than unknowns, it will admit solutions only when some relations between the coefficients hold. We now proceed to show how this happens. Combining eq. (4.12) and eq. (4.15) on one side, eq. (4.12) and eq. (4.18) on the other side, one is led to the relations

$$\tau_{23}=j\,\alpha\,\mu_{23}+\frac{\tau_{12}}{-i+\mu_{12}}\,,\qquad \mu_{31}=-i\frac{(-i+\mu_{12})(j\,\alpha-\tau_{31})}{\tau_{12}}\,. \tag{4.21}$$

In the same way, from eq. (4.13) , eq. (4.16) , and eq. (4.19), one finds

$$\mu_{23}=-i\frac{\tau_{12}}{j\,\alpha\,(1+\mu_{12}^2)}\,,\qquad \tau_{31}=-i\frac{j\alpha}{\mu_{12}}\,. \tag{4.22}$$

From these formula one finds the relation displayed in eq. (4.5) which was shown to be necessary for the system to admit a solution. Subtracting eq. (4.20) from eq. (4.17) , one has

$$\sigma_n\equiv\frac{Y_n}{Y_{n-1}}=i\,j\,\sqrt{\frac{1+j-n}{j+n}}\,\frac{\alpha(i+\mu_{12})}{\tau_{12}}\,. \tag{4.23}$$

Using this result back in eq. (4.17) one obtains

$$\tau_{12}=j\,\alpha\,\sqrt{1+\mu_{12}^2}\,. \tag{4.24}$$

These expressions transform eq. (4.14) into an identity. So, the solution to our system of equations depends on the sole complex number μ_{12}; the other coefficients μ_{jk}, τ_{jk} as well as the components of the state $|Y\rangle$.

We switch to a new variable ζ defined by the ratio of the last two components of our state:

$$\frac{1}{\zeta} \equiv \frac{Y_j}{Y_{j-1}} = -\frac{1}{\sqrt{2j}} \frac{1 - i\,\mu_{12}}{\sqrt{1+\mu_{12}^2}} \Longleftrightarrow \mu_{12} = i\frac{2j-\zeta^2}{2j+\zeta^2}\,. \tag{4.25}$$

The coefficients of the set of equations now assume the form:

$$\begin{aligned}
\mu_{23} &= \frac{i}{2\sqrt{2\,j}}\,\frac{2j+\zeta^2}{\zeta}\,, & \mu_{31} &= -2\sqrt{2\,j}\,\frac{\zeta}{-2j+\zeta^2}\,, & \tau_{12} &= -\frac{2\sqrt{2}\,j}{\sqrt{1+j}}\,\frac{\zeta}{2j+\zeta^2}\,,\\
\tau_{23} &= -\frac{i}{2\sqrt{2}\,\sqrt{1+j}}\,\frac{2j-\zeta^2}{\zeta}\,, & \tau_{31} &= -\sqrt{\frac{j}{j+1}\,\frac{2j+\zeta^2}{2j-\zeta^2}}\,.
\end{aligned} \tag{4.26}$$

Concerning the components of the state $|Y\rangle$, one has

$$\begin{aligned}
&\frac{Y_{-j+1}}{Y_{-j}} = \frac{2j}{\zeta}\,, \qquad \frac{Y_j}{Y_{j-1}} = \frac{1}{\zeta}\,, \qquad \frac{Y_n}{Y_{n-1}} = \sqrt{2j}\,\sqrt{\frac{1+j-n}{j+n}}\,\frac{1}{\zeta}\,,\\
&\text{when } \; -j+1 \leq n \leq j-1\,.
\end{aligned} \tag{4.27}$$

By a recursive reasoning, one can express all the components in terms of the highest one

$$Y_m = \frac{1}{(2j)^{\frac{1}{2}(j-m)}} \begin{pmatrix} 2j \\ j-m \end{pmatrix}^{\frac{1}{2}} \zeta^{j-m}\, Y_j\,. \tag{4.28}$$

The normalization of the wave function results in the condition

$$Y_j^2 \left(\frac{\zeta\bar{\zeta}}{2j}\right)^j \sum_{m=-j}^{j} \begin{pmatrix} 2j \\ j-m \end{pmatrix} \left(\frac{2j}{\zeta\bar{\zeta}}\right)^m = 1\,. \tag{4.29}$$

Rewriting this sum in terms of the index $n = m + j$ which goes from 0 to $2j$, one obtains

$$Y_j = \left(\frac{2j}{2j+\zeta\bar{\zeta}}\right)^j\,. \tag{4.30}$$

Replacing this in the expression of Y_m and denoting from now on the state by the complex parameter ζ, one obtains

$$|\zeta\rangle = \frac{1}{(2j+\zeta\bar{\zeta})^j} \sum_{m=-j}^{j} R_m\, \zeta^{-m+j}\, |m\rangle\,, \tag{4.31}$$

where the real constants R_m are given by

$$R_m = (2j)^{\frac{1}{2}(m+j)} \begin{pmatrix} 2j \\ j-m \end{pmatrix}^{1/2}$$

for all n since

$$R_{-j} = 1 \qquad \text{and} \qquad R_j = (2j)^j \,. \tag{4.32}$$

We see that considering the uncertainty relation we derived in the third section, which is weaker than Heisenberg's, we pass from a situation in which no state saturates all the paired bounds to one in which there is a family of states achieving that. This family is parameterized by one complex number ζ, like the generalized coherent states on the fuzzy sphere obtained in [6]. The difference between the two works is twofold. Firstly, the starting point here is the uncertainty relation. Secondly, the Fock space in which we work is finite dimensional so that we don't need a truncature procedure.

After some algebra, one finds for the scalar product the expression

$$\langle\zeta|\eta\rangle = \frac{(2j+\bar{\zeta}\eta)^{2j}}{(2j+\bar{\zeta}\zeta)^j(2j+\bar{\eta}\eta)^j} \,. \tag{4.33}$$

Using the formula

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x = e^a, \tag{4.34}$$

one obtains in the limit $j \to \infty$ the formula

$$|\langle\eta|\zeta\rangle|^2 = e^{-|\eta-\zeta|^2} \,. \tag{4.35}$$

which is also valid for the ordinary coherent states on the non commutative plane. It has been known that there is a limit in the parameter space of the fuzzy sphere which reproduces the non commutative plane [35]. The result obtained here goes in the same direction: the two structures have things in common.

However, contrary to the non commutative plane, for any fixed generalized squeezed state $|\zeta\rangle$, there is another state $|\eta\rangle$ to which it is orthogonal; it satisfies $\zeta\bar{\eta} = -2j$. However, it disappears in the commutative limit.

5. The stereographic projection

We now show how the states built so far reproduce the stereographic projection. The mean value $\langle x_1\rangle$ can be written as

$$\langle\zeta|\hat{x}_1|\zeta\rangle = \frac{1}{\sqrt{j(j+1)}}\frac{\zeta^j\bar{\zeta}^j}{(2j+\zeta\bar{\zeta})^{2j}}\sum_{m,n} R_m R_n \zeta^{-m}\bar{\zeta}^{-n}\langle n|J_1|m\rangle \,. \tag{5.1}$$

Expressing J_1 as in eq. (4.9), we can split the sum in the previous equation

$$\sum_{m,n} R_m R_n \zeta^{-m}\bar{\zeta}^{-n}\langle n|J_1|m\rangle = I_1 + I_2 \,. \tag{5.2}$$

The first term can be computed as follows

$$I_1 = \sum_{m=-j}^{j-1} R_m R_{m+1}\zeta^{-m}\bar{\zeta}^{-m-1}\sqrt{(j+m+1)(j-m)}$$

$$= \zeta \sum_{m=-j}^{j-1} (2j)^{m+j+3/2} \binom{2j-1}{j+m} (\zeta\bar{\zeta})^{-m-1}$$

$$= (2j)^{j+3/2} \frac{1}{\bar{\zeta}} \sum_{n=0}^{2j-1} \binom{2j-1}{n} \left(\frac{2j}{\zeta\bar{\zeta}}\right)^{n-j}$$

$$= (2j)^{3/2}\, \zeta \,\frac{(2j+\zeta\bar{\zeta})^{2j-1}}{(\zeta\bar{\zeta})^{j}} \,. \tag{5.3}$$

Similarly,

$$I_2 = \sum_{m=-j+1}^{j} R_m R_{m-1} \zeta^{-m} \bar{\zeta}^{-m+1} \sqrt{(j-m+1)(j+m)}$$

$$= (2j)^{3/2}\, \bar{\zeta}\, \frac{(2j+\zeta\bar{\zeta})^{2j-1}}{(\zeta\bar{\zeta})^{j}} \,, \tag{5.4}$$

so that the wanted mean value rakes the form

$$\langle\zeta|\hat{x}_1|\zeta\rangle = \frac{\sqrt{2}j}{\sqrt{j+1}} \frac{\zeta+\bar{\zeta}}{2j+\zeta\bar{\zeta}} \,. \tag{5.5}$$

The two remaining quantities $\langle\hat{x}_k\rangle$ can be obtained using the fact that eq. (4.1) implies $\langle\hat{m}_{jk}\rangle = 0$. For example, the simplification

$$\tau_{12} + \langle\zeta|\hat{x}_1|\zeta\rangle = \frac{\sqrt{2}j}{\sqrt{j+1}} \left(-\frac{2\zeta}{2j+\zeta^2} + \frac{\zeta+\bar{\zeta}}{2j+\zeta\bar{\zeta}} \right)$$

$$= \frac{\sqrt{2}j}{\sqrt{j+1}} \frac{(\bar{\zeta}-\zeta)(2j-\zeta^2)}{(2j+\zeta^2)(2j+\zeta\bar{\zeta})} \tag{5.6}$$

allows one to infer from $\langle\hat{m}_{12}\rangle = 0$ the mean value

$$\langle\zeta|\hat{x}_2|\zeta\rangle = -\frac{1}{\mu_{12}}(\tau_{12} + \langle\zeta|\hat{x}_1|\zeta\rangle) = i\, \frac{\sqrt{2}j}{\sqrt{j+1}} \frac{\bar{\zeta}-\zeta}{2j+\zeta\bar{\zeta}} \,. \tag{5.7}$$

In the same way,

$$\langle\zeta|\hat{x}_3|\zeta\rangle = -\left(\langle\zeta|\hat{x}_1|\zeta\rangle + \frac{\tau_{31}}{\mu_{31}} \right) \mu_{31} = -\sqrt{\frac{j}{j+1}} \frac{-2j+\zeta\bar{\zeta}}{2j+\zeta\bar{\zeta}} \,, \tag{5.8}$$

while the vanishing of $\langle\hat{m}_{23}\rangle$ results in an identity.

The expressions of these mean values resemble the ones defining the stereographic projection. The link can be made more transparent if one switches to the variable

$$\zeta = \sqrt{2j}\beta \tag{5.9}$$

since the mean values now read

$$\langle\hat{x}_1\rangle = \sqrt{\frac{j}{j+1}} \frac{\beta+\bar{\beta}}{1+\beta\bar{\beta}} \,, \qquad \langle\hat{x}_2\rangle = i\sqrt{\frac{j}{j+1}} \frac{\bar{\beta}-\beta}{1+\beta\bar{\beta}} \,, \qquad \langle\hat{x}_3\rangle = \sqrt{\frac{j}{j+1}} \frac{\beta\bar{\beta}-1}{1+\beta\bar{\beta}} \,. \tag{5.10}$$

The stereographic map is then recovered in the large j limit. Concerning the dispersions, one has

$$1 = \langle \hat{x}_1^2 \rangle + \langle \hat{x}_2^2 \rangle + \langle \hat{x}_3^2 \rangle \,, \qquad \frac{j}{j+1} = \langle \hat{x}_1 \rangle^2 + \langle \hat{x}_2 \rangle^2 + \langle \hat{x}_3 \rangle^2 \,,$$
$$\Longrightarrow (\Delta x_1)^2 + (\Delta x_2)^2 + (\Delta x_3)^2 = \frac{1}{j+1} \tag{5.11}$$

so that in the large j limit all the three Δx_k vanish.

In terms of this new variable, the generalized squeezed state reads

$$|\beta\rangle = \left(\frac{1}{1+\beta\bar{\beta}}\right)^j \sum_{m=-j}^{j} \begin{pmatrix} 2j \\ j-m \end{pmatrix}^{1/2} \beta^{-m+j} |m\rangle \,, \tag{5.12}$$

while the scalar product takes the form

$$\langle \gamma|\beta\rangle = \left(\frac{(1+\bar{\gamma}\beta)^2}{(1+\gamma\bar{\gamma})(1+\beta\bar{\beta})}\right)^j . \tag{5.13}$$

We can compute these mean values in a different way. For example, from eq. (3.7) we have

$$\langle \hat{p} \rangle = -\frac{\bar{\lambda}_4}{\bar{\lambda}_2} . \tag{5.14}$$

Considering the equation $\hat{m}_{12}|\zeta\rangle = 0$, we have to replace $\hat{x}$ by $\hat{x}_1$ and $\hat{p}$ by $\hat{x}_2$. The separation between real and imaginary parts $\mu_{12} = \bar{\lambda}_1 + i\bar{\lambda}_2$, $\tau_{12} = \bar{\lambda}_3 + i\bar{\lambda}_4$ reads

$$\bar{\lambda}_2 = \frac{(2j+\zeta\bar{\zeta})(2j-\zeta\bar{\zeta})}{(2j+\zeta^2)(2j+\bar{\zeta}^2)} \,, \qquad \bar{\lambda}_4 = -\frac{2\sqrt{2}j}{\sqrt{1+j^2}} \frac{\zeta_2(2j-\zeta\bar{\zeta})}{(2j+\zeta^2)(2j+\bar{\zeta}^2)} \,, \tag{5.15}$$

ζ_2 being the imaginary part of ζ. Replacing this into eq. (5.14) appropriately modified, one recovers eq. (5.7).

We shall finish this section with the simplest possible illustration, i.e the case $j = 1/2$. The variables ζ and β then coincide. The coefficients of the operators $\hat{m}_{jk}$ have the form

$$\begin{aligned} \mu_{1,2} &= i\frac{1-\zeta^2}{1+\zeta^2} \,, & \tau_{1,2} &= -\frac{2}{\sqrt{3}}\frac{\zeta}{1+\zeta^2} \,, & \mu_{2,3} &= i\frac{1+\zeta^2}{2\zeta} \,, \\ \tau_{2,3} &= -i\frac{1}{2\sqrt{3}}\frac{1-\zeta^2}{\zeta} \,, & \mu_{3,1} &= \frac{2\zeta}{1-\zeta^2} \,, & \tau_{3,1} &= -\frac{1}{\sqrt{3}}\frac{1+\zeta^2}{1-\zeta^2} \,, \end{aligned} \tag{5.16}$$

while the generalized squeezed states read

$$|\zeta\rangle = \frac{1}{\sqrt{1+|\zeta|^2}} \left(\zeta \left| -\frac{1}{2} \right\rangle + \left| \frac{1}{2} \right\rangle \right) . \tag{5.17}$$

The deformed stereographic projection takes the form

$$\langle \hat{x}_1 \rangle = \frac{1}{\sqrt{3}} \frac{\bar{\zeta}+\zeta}{1+\zeta\bar{\zeta}} \,, \qquad \langle \hat{x}_2 \rangle = \frac{i}{\sqrt{3}} \frac{\bar{\zeta}-\zeta}{1+\zeta\bar{\zeta}} \,, \qquad \langle \hat{x}_3 \rangle = \frac{1}{\sqrt{3}} \frac{1-\bar{\zeta}\zeta}{1+\zeta\bar{\zeta}} . \tag{5.18}$$

JHEP05(2004)052

6. Conclusions

We have derived an uncertainty relation which is weaker than the Heisenberg's. On the fuzzy sphere, this relation can be saturated for all the pairs of variables. The states built in this way possess some interesting properties. They are parameterized by a complex number and realize the stereographic map. This fact is promising in the sense that the star product associated to these states will naturally be defined on functions of two real variables i.e exactly the geometric dimension of the sphere [27].

The link between our states and the geometry of the sphere suggest the possibility of studying Q.F.T in this context using a star product defined on the stereographic coordinates. An important question will then be the properties of field theory on the fuzzy sphere [36] which are recovered in such a formulation. A study of classical solutions [37, 38] would of course be interesting.

A. High dimensional extension

There exists a model possessing a minimal uncertainty in length while preserving rotational and translational symmetries [33]. Its non trivial commutation relations are:

$$[\hat{x}_j, \hat{p}_k] = i\hbar \left(f(\hat{\vec{p}}^2)\delta_{jk} + g(\hat{\vec{p}}^2)\hat{p}_j\hat{p}_k \right) , \tag{A.1}$$

supplemented by the condition

$$g = \frac{2ff'}{f - 2p^2 f'} . \tag{A.2}$$

The functions f and g are supposed positive. This theory admits four couples of non commuting variables [7]: $(x_1, p_1), (x_2, p_2), (x_1, p_2)$ and (x_2, p_1). It was shown in [7] that one can not saturate the associated non trivial uncertainties simultaneously. The same argument can be repeated here to show this is still the case even for the enlarged uncertainties we derived. How can one define squeezed states which do not discriminate between the variables? Although we will not apply it explicitly, we think one may be led to look at the structure of uncertainty relations implying more than two operators. We outline this below.

In the second section, we have seen that in $2D$ an analysis based on an operator leads to an uncertainty implying the mean values of at most quadratic expressions of the variables. The question we would like to answer in this section is the following: what are the general uncertainties implying more than two variables?

We now go one step further and consider three variables simultaneously. Mimicking what was done in the second section, we consider an operator of the form

$$\hat{\Theta}_\lambda = \hat{x}_1 + (\lambda_1 + i\lambda_2)\hat{x}_2 + (\lambda_3 + i\lambda_4)\hat{p}_1 + (\lambda_5 + i\lambda_6)I . \tag{A.3}$$

We build the function $\varphi(\vec{\lambda})$ similarly to what was done in eq. (3.3):

$$\begin{aligned}\varphi(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) = {} & \langle \hat{x}_1^2 \rangle + \lambda_1 A_{x_1,x_2} + \lambda_2 C_{x_1,x_2} + \lambda_3 A_{x1,p1} + \lambda_4 C_{x_1,p_1} + \\ & +2\lambda_5\langle \hat{x}_1 \rangle + (\lambda_1^2 + \lambda_2^2)\langle \hat{x}_2^2 \rangle + (\lambda_1\lambda_3 + \lambda_2\lambda_4) A_{x_2,p_1} +\end{aligned}$$

$$+(\lambda_1\lambda_4-\lambda_2\lambda_3)C_{x_2,p_1}+2(\lambda_1\lambda_5+\lambda_2\lambda_6)\langle\hat{x}_2\rangle+(\lambda_3^2+\lambda_4^2)\langle\hat{p}_1^2\rangle+$$
$$+2(\lambda_3\lambda_5+\lambda_4\lambda_6)\langle\hat{p}_1\rangle+(\lambda_5^2+\lambda_6^2)\,. \tag{A.4}$$

We can diagonalize it, similarly to what was done in eq. (3.5):

$$\begin{aligned}\varphi(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6) =\; &\sigma_1\,(\lambda_1+\mu_{12}\,\lambda_2+\mu_{13}\,\lambda_3+\mu_{14}\,\lambda_4+\mu_{15}\,\lambda_5+\mu_{16}\,\lambda_6+\mu_{10})^2+\\ &+\sigma_2\,(\lambda_2+\mu_{23}\,\lambda_3+\mu_{24}\,\lambda_4+\mu_{25}\,\lambda_5+\mu_{26}\,\lambda_6+\mu_{20})^2+\\ &+\sigma_3\,(\lambda_3+\mu_{34}\,\lambda_4+\mu_{35}\,\lambda_5+\mu_{36}\,\lambda_6+\mu_{30})^2+\\ &+\sigma_4\,(\lambda_4+\mu_{45}\,\lambda_5+\mu_{46}\,\lambda_6+\mu_{40})^2+\\ &+\sigma_5\,(\lambda_5+\mu_{56}\,\lambda_6+\mu_{50})^2+\\ &+\sigma_6\,(\lambda_6+\mu_{60})^2+\\ &+\sigma_7\,.\end{aligned} \tag{A.5}$$

The coefficients σ_i and $\mu_{j,k}$(which have nothing to do with those of section 4) are easily found:

$$\begin{aligned}\sigma_1 &= \langle\hat{x}_2^2\rangle\,, \qquad \mu_{12}=0\,, \qquad \mu_{13}=\frac{A_{x_2,p_1}}{2\langle\hat{x}_2^2\rangle}\,, \qquad \mu_{14}=\frac{C_{x_2,p_1}}{2\langle\hat{x}_2^2\rangle}\,, \qquad \mu_{15}=\frac{\langle\hat{x}_2\rangle}{\langle\hat{x}_2^2\rangle}\,,\\ \mu_{16} &= 0\,, \qquad \mu_{10}=\frac{A_{x_1,x_2}}{2\langle\hat{x}_2^2\rangle}\,, && \text{(A.6)}\\ \sigma_2 &= \sigma_1\,, \qquad \mu_{23}=-\mu_{14}\,, \qquad \mu_{24}=\mu_{13}\,, \qquad \mu_{25}=0\,, \qquad \mu_{26}=\mu_{15}\,,\\ \mu_{20} &= \frac{C_{x_1,x_2}}{2\langle\hat{x}_2^2\rangle}\,, && \text{(A.7)}\\ \sigma_3 &= \frac{-A_{x_2,p_1}^2-C_{x_2,p_1}^2+4\langle\hat{p}_1^2\rangle\langle\hat{x}_2^2\rangle}{4\langle\hat{x}_2^2\rangle}\,, \qquad \mu_{34}=0\,,\\ \mu_{35} &= \frac{2(-2\langle\hat{p}_1\rangle\langle\hat{x}_2^2\rangle+A_{x_2,p_1}\langle\hat{x}_2\rangle)}{A_{x_2,p_1}^2+C_{x_2,p_1}^2-4\langle\hat{p}_1^2\rangle\langle\hat{x}_2^2\rangle}\,, \qquad \mu_{36}=-\frac{2\,C_{x2,p1}\,\langle\hat{x}_2\rangle}{A_{x2,p1}^2+C_{x2,p1}^2-4\,\langle\hat{p}_1^2\rangle\langle\hat{x}_2^2\rangle}\,,\\ \mu_{30} &= \frac{A_{x1,x2}A_{x2,p1}-C_{x1,x2}C_{x2,p1}-2\,A_{x1,p1}\,\langle\hat{x}_2^2\rangle}{A_{x2,p1}^2+C_{x2,p1}^2-4\,\langle\hat{p}_1^2\rangle,\langle\hat{x}_2^2\rangle}\,, && \text{(A.8)}\\ \sigma_4 &= \sigma_3\,, \qquad \mu_{45}=-\mu_{36}\,, \qquad \mu_{46}=\mu_{35}\,,\\ \mu_{40} &= \frac{A_{x2,p1}C_{x1,x2}+A_{x1,x2}C_{x2,p1}-2\,C_{x1,p1}\,\langle\hat{x}_2^2\rangle}{A_{x2,p1}^2+C_{x2,p1}^2-4\,\langle\hat{p}_1^2\rangle\langle\hat{x}_2^2\rangle}\,, && \text{(A.9)}\\ \sigma_5 &= 1-\mu_{15}^2\,\sigma_1-(\mu_{35}^2+\mu_{45}^2)\,\sigma_3\,, \qquad \mu_{56}=0\\ \mu_{50} &= \frac{1}{\sigma_5}\left[\langle\hat{x}_1\rangle-\mu_{10}\,\mu_{15}\,\sigma_1-(\mu_{30}\,\mu_{35}-\mu_{36}\,\mu_{40})\,\sigma_3\right], && \text{(A.10)}\\ \sigma_6 &= \sigma_5\,, \qquad \mu_{60}=-\frac{1}{\sigma_5}\left[\mu_{15}\mu_{20}\sigma_1+(\mu_{36}\mu_{30}+\mu_{35}\mu_{40})\sigma_3\right], && \text{(A.11)}\\ \sigma_7 &= \langle\hat{x}_1^2\rangle-(\mu_{10}^2+\mu_{20}^2)\sigma_1-(\mu_{30}^2+\mu_{40}^2)\sigma_3-(\mu_{50}^2+\mu_{60}^2)\sigma_5\,. && \text{(A.12)}\end{aligned}$$

As in the previous section, one has the inequalities

$$\sigma_1,\ldots,\sigma_6>0\,, \qquad \sigma_7\geq 0\,. \tag{A.13}$$

For a fixed state $|\psi\rangle$, the minimum of the function φ is attained for special values $\bar{\lambda}_1, \ldots, \bar{\lambda}_6$ which can be expressed in function of the observables of the states. For example,

$$\bar{\lambda}_6 = -\mu_{60} , \qquad \bar{\lambda}_5 = -\mu_{50} - \mu_{56}\bar{\lambda}_6 , \qquad \bar{\lambda}_4 = -\mu_{40} - \mu_{45}\bar{\lambda}_5 - \mu_{46}\bar{\lambda}_6 , \tag{A.14}$$

where the quantities μ_{jk} are given in eqs. (A.6), (A.7), (A.8), (A.9), (A.10), (A.11), (A.12).

We are now in a position to characterize the kernel of the operator $\hat{\Theta}_{\bar{\lambda}}$ of eq. (A.3). For such elements, the function φ attains its minimum so that the parameters of the operator are linked to observables by the relations displayed above. As is equals zero, one has in addition that

$$\sigma_7 = 0 . \tag{A.15}$$

We have obtained an uncertainty mixing the three coordinates and which can be saturated.

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JHEP05(2004)052

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